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On the base space of a semi-universal deformation of rational quadruple points

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Introduction

This paper is devoted to the study of a particular class of normal surface singularities: the rational quadruple points. We will determine the base space of a semi-universal deformation of such a singularity. The answer turns out to be unexpectedly simple: the isomorphism type of the base space of a rational quadruple point is completely determined by two numbers, s and n. The base space then is isomorphic to $S \times B(n)$, where S is a smooth germ of dimension s and where $B(n)$ is a certain "universal" space defined in (3.5). A rational quadruple point with the star-shaped resolution graph (shown in Figure 1) has the factor $B(n)$ in its space. We call such a singularity an n-star.

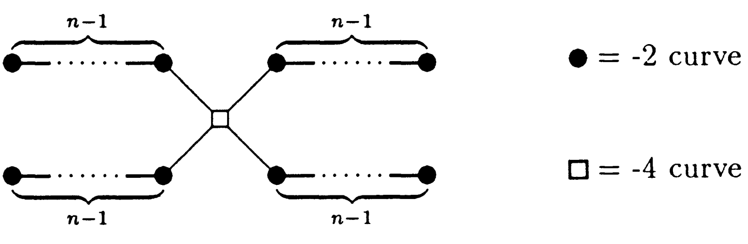

FIGURE 1

In general there are several approaches to finding the semi-universal deformation of a (normal surface) singularity $\tilde{X}$. In the first place there is the *direct method*: one starts with the set of equations defining $\tilde{X}$ as embedded in $\mathbf{C}^N$ for some large N, and then one just computes. For this to work in practice the equations must have a sufficiently strong structure. For example, rational triple points (see [Tj]) (Cohen-Macaulay codimension 2), the cone over the rational normal curve of degree n (see [Pi]), n lines in $\mathbf{C}^n$, etc., can be handled in this way. It seems however that the equations for the rational quadruple points are not known sufficiently well to compute the base spaces for them in this way.

Secondly, there is the method of (*partial*) *resolutions*. Here one starts with a (partial) resolution Y of $\tilde{X}$ and then studies the deformation theory of Y (which is usually much simpler) and finally one tries to blow down the deformed Y to get a deformation of $\tilde{X}$. This method works quite well for obtaining information on the components of the base space for rational singularities. For example, all deformations of a resolution of $\tilde{X}$ can be blown down and give rise to the so-called *Artin component* of (the base space of) $\tilde{X}$ (see [Wa]). Recently, Kollár and Shepherd-Barron [K-S] developed a method by which one can, for instance, determine the number of components in the base space of a cyclic quotient singularity. (From their approach it is also clear that the n-star singularity has (at least) $n + 1$ components in its base space.) However, the list of resolution graphs of rational quadruple points is quite long and contains many "exceptional" graphs, so this method seems to be quite involved. Furthermore, it does not really lead to equations for the base spaces.

We propose to use a different method: the method of *projections*, which we will explain now. One starts with $\tilde{X}$ embedded in $\mathbf{C}^N$ for some large N, and then projects $\tilde{X}$ generically into $\mathbf{C}^3$. The image X then will have a curve Σ as double locus. In such a situation the authors introduced in [J-S1] and [J-S2] a deformation functor, $\mathrm{Def}(\Sigma, X)$, which we called the functor of *admissible deformations* of Σ and X. We recall the definition. Let $\mathbf{C}$ be a category of spaces (e.g., those of germs of analytic spaces). A diagram of spaces $\Sigma_S \hookrightarrow X_S$, flat over some base space S, is called *admissible* if and only if $\Sigma_S \hookrightarrow \mathscr{C}_{X_S/S}$, where $\mathscr{C}_{X_S/S}$ is the relative critical space as defined by Teissier [Te, p. 587]. Now let $\Sigma \hookrightarrow X$ be an admissible diagram over the spectrum of the ground field. Then the *functor of admissible deformations*, $\mathrm{Def}(\Sigma, X)$: $\mathbf{C} \to \mathbf{Set}$ is defined by:

$$S \to \{\text{isomorphism classes of deformations of } \Sigma \hookrightarrow X \text{ over } S \text{ which are admissible}\}.$$

We recall the main result of the paper [J-S3]; see also [J-S1]. Let $X \subset \mathbf{C}^3$ be a surface singularity with an ordinary double curve Σ as reduced singular locus. Let $\tilde{X} \to X$ be the normalization of X. Then one has a natural equivalence of functors:

$$\mathrm{Def}(\Sigma, X) \tilde{\to} \mathrm{Def}(\tilde{X} \to X).$$

Here, $\mathrm{Def}(\tilde{X} \to X)$ denotes the deformation functor of the diagram $\tilde{X} \to X$ (cf. [Bu]). Moreover, the natural forgetful map: $\mathrm{Def}(\tilde{X} \to X) \to \mathrm{Def}(\tilde{X})$ is smooth [J-S3, (1.3), (1.4)]. Therefore, by these results, we have that the base space of admissible deformations of $\Sigma \hookrightarrow X$ is up to a smooth factor the same as the base space of $\tilde{X}$. Now, essentially because Σ is Cohen-Macaulay of codimension 2 and X is given by one equation, this is much more "computable" than if we work directly with the equations for $\tilde{X}$. At first sight it seems that this method has two serious drawbacks. In the first place one has to choose a *generic* projection (to get an ordinary double curve), and naturally given projections usually are not generic. In the second place it is quite hard to find the explicit equation for X. For rational triple points, it is already a lot of work to write down explicit equations for X corresponding to the different resolution graphs and, for quadruple points, it becomes quite hopeless. We only give one example of our (very incomplete) list. (It appears convenient to use the theory of *limits* (see [Str]) to obtain equations for singularities that come in series.)

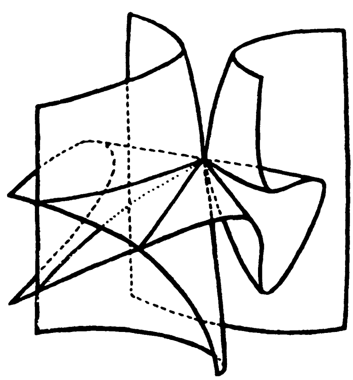

FIGURE 2

Example. For the equation

$$f = (x - y) \cdot \left((x + y) \cdot (z^2 + xy^2) + (x - y)^{k+1} \cdot y^2\right) + z^l \cdot (z^2 + xy^2)^2$$

we have the qualitative picture of $X_{\mathbf{R}} := \{(x, y, z) \in \mathbf{R}^3 | f(x, y, z) = 0\}$ as shown in Figure 2. The resolution graph of the normalization $\tilde{X}$ is shown in Figure 3.

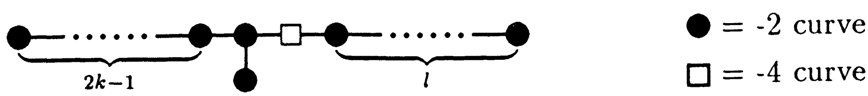

FIGURE 3

It turns out, however, that when we are interested in determining base spaces *up to smooth factors*, both drawbacks mentioned can be turned into advantages. The idea is the following: two weakly normal surface singularities in $\mathbf{C}^3$, X_1 and X_2, with (reduced) singular loci Σ_1 and Σ_2, respectively, have isomorphic base spaces (up to a smooth factor) for their semi-universal admissible deformation if:

(1) X_1 and X_2 have isomorphic normalization $\tilde{X}$. As $\tilde{X}$ will have many different projections into $\mathbf{C}^3$, we get many weakly normal surfaces with (up to a smooth factor) isomorphic base spaces.

(2) $\Sigma_1 = \Sigma_2$ and X_1 is *I^2-equivalent* to X_2. Recall that we call two surfaces X_1 and $X_2 \subset \mathbf{C}^3$ (with the same singular locus Σ, defined by an ideal I) I^2-equivalent if there are defining functions f_1 and f_2 for X_1 and X_2, resp., such that $f_1 - f_2 \in I^2$ (see [J-S1], [J-S2]).

The fact that I^2-equivalent X_1 and X_2 have (up to a smooth factor) isomorphic base spaces for their semi-universal admissible deformation is, in the authors' opinion, a simple but important result [J-S2, (1.16)]. One could say that in this sense $\mathrm{Def}(\Sigma, X)$ depends more on Σ than on X.

So we have two principles that can be used to determine the base space of a semi-universal deformation of a weakly normal surface singularity up to a smooth factor. We can even take the "transitive hull" of these two principles, making it into a powerful tool.

It turns out in Section 2 that these principles are strong enough to determine the base space of a semi-universal admissible deformation of a weakly normal surface with reduced singular locus a curve in $\mathbf{C}^3$ of multiplicity three and Gorenstein type two. In Section 1 we prove that rational quadruple points have a generic projection such that the reduced singular locus is a multiplicity-three and type-two curve, and prove some facts about these curves. In Section 3 we compute a semi-universal deformation of an n-star, thereby getting equations for the space $B(n)$ that was mentioned in the beginning of the Introduction. Finally, in Section 4 we study the structure of the space $B(n)$. It is proved that $B(n)$ has $n+1$ irreducible components, of which the normalizations are smooth.

Conventions. We will work in the category of analytic spaces, but as we work almost exclusively with germs, we do not make notational distinction between germs and suitable representatives. $\tilde{X}$ will always be a germ of a normal surface singularity and X a weakly normal surface in $\mathbf{C}^3$. The reduced singular locus of X is denoted by Σ, and I will be the ideal of functions in $\mathbf{C}\{x, y, z\}$ which vanish on Σ. The defining function f of such an X will be an element of $\int I$, the ideal of functions in I whose partial derivatives are also in I.

As I is reduced, $\int I$ is just the second symbolic power of I. The (Gorenstein) *type* of a germ is the number of generators of the dualizing module.

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1. The singular locus of a generic projection of rational quadruple points

The main idea of this paper is to bring questions about deformations of normal surface singularities back to the study of the singular locus of a "generic" projection of such a singularity. So, in order to study rational quadruple points, we have to study which curve singularities appear as a singular locus when we project a rational quadruple point. We start with some general numerical relations related to a generic linear projection.

LEMMA 1.1. *Let $\tilde{X} \subset \mathbf{C}^n$ be a ((multi-) germ of a) normal surface singularity, where $N = \mathrm{Embdim}(\tilde{X})$ is the embedding dimension of $\tilde{X}$. Let $L: \mathbf{C}^N \to \mathbf{C}^3$ be a generic linear projection and let $X = L(\tilde{X}) \subset \mathbf{C}^3$ be its image. Let Σ be the reduced singular locus of X and let $\tilde{H}$ and H be the generic hyperplane sections of $\tilde{X}$ and X, respectively. Then*:

(i) $m := \mathrm{Mult}(\tilde{X}) = \mathrm{Mult}(X) = \mathrm{Mult}(\tilde{H}) = \mathrm{Mult}(H) \geq N - 1$.

(ii) $\mathrm{Mult}(\Sigma) = \delta(H) - \delta(\tilde{H})$.

(iii) $\delta(H) \geq m \cdot (m-1)/2$; $\delta(\tilde{H}) \geq m - 1$.

(iv) $\mathrm{type}(\Sigma) \geq N - 3$.

Proof. (i) is obvious because we have a linear projection. The inequality expresses the minimality of the embedding of $\tilde{X}$ in $\mathbf{C}^N$. Statement (ii) follows when we move the hyperplane H away from the special point. We then get as intersection with X a curve with $\mathrm{Mult}(\Sigma)$ ordinary double points. But the jump in δ in a family of curves is equal to the δ of the special fibre of the normalization of the family (see [L-L-T]); so in this case it is equal to $\delta(\tilde{H})$. Statement (iii) is a generality: given the embedding dimension and the multiplicity of the curve, one has a lower bound for its δ-invariant, which is in the stated cases as above. (For a proof, see [B-C, 3.3].) Statement (iv) comes from the following: Σ is Cohen-Macaulay of codimension 2, so the equations for Σ are obtained as the maximal minors of a $t \times (t+1)$ matrix. Then type$(\Sigma) = t$. As in [J-S3, (3.1)] this gives us an embedding of $\tilde{X}$ into a smooth space of dimension $3 + t$; hence $N \leq t + 3$. ⊠

Lemma 1.2. *If $\tilde{X}$ is a germ of a rational surface singularity, then all the inequalities of Lemma* 1.1 *are in fact equalities.*

Proof. This lemma is a reflection of the strong minimality properties enjoyed by rational singularities. For the fact that $N = m + 1$ we refer to [Ar]. For the statement that $\delta(\tilde{H}) = m - 1$, see [B-C, 4.1.2, 3.3]. Curves with $\delta(\tilde{H}) = m - 1$ are exactly the so-called partition curves [B-C] for which one easily sees that a generic projection H into $\mathbf{C}^3$ has $\delta(H) = m(m-1)/2$. In particular we get mult$(\Sigma) = (m-1)(m-2)/2$. The statement about the type can be seen as follows: because Σ is Cohen-Macaulay, the subscheme of $\mathbf{C}^2$ given by $\Sigma \cap H$ has length $(m-1)\cdot(m-2)/2$ and, by Lemma 1.1, type$(\Sigma \cap H) \geq m - 2$. From these facts alone it already follows that the ideal of $\Sigma \cap H$ is the ideal $\mathfrak{m}^{m-1}$, where $\mathfrak{m}$ is the maximal ideal of $\mathbf{C}\{y, z\} = \mathcal{O}_{\mathbf{C}^2,0}$. Hence indeed type$(\Sigma) =$ type$(\Sigma \cap H) = m - 2$. ⊠

Corollary 1.3. *$\tilde{X}$ rational triple point $\Rightarrow$ Σ is smooth*; i.e., *X is a line singularity. $\tilde{X}$ rational quadruple point $\Rightarrow$ Σ has multiplicity three and type two.*

Proof is immediate from Lemma 1.2. ⊠

Lemma/*Definition* 1.4. *Let Σ be a Cohen-Macaulay curve germ of multiplicity three and type two. Then the equations for Σ can be obtained as the* 2×2*-minors of the following matrix*:

$$M = \begin{pmatrix} y & z + a & b \\ c & y + d & z \end{pmatrix}.$$

Here a, b, c and d are functions only depending on x. The λ-invariant of such a curve is defined as:

$$\lambda(\Sigma) := \min(\operatorname{ord}(a), \operatorname{ord}(b), \operatorname{ord}(c), \operatorname{ord}(d)).$$

Conversely, if $\lambda(\Sigma) \geq 1$, then the minors of the above matrix do define a Cohen-Macaulay curve germ of multiplicity three and type two.

Proof. Choose a generic projection of Σ on a line with coordinate x. Then Σ can be considered as the total space of a flat deformation of Σ intersected with $x = 0$. This subscheme of $\mathbf{C}^2$ is defined by $\mathfrak{m}^2 = (y^2, yz, z^2)$. As these equations can be obtained from the matrix as above (with $a = b = c = d = 0$), we find the indicated form for the equations of Σ. ⊠

Remark 1.5. Curves of multiplicity three can be classified and J. Stevens has sent us the complete list. However, it turns out to be possible to pursue our arguments without going into the fine structure of this classification. Note that a Cohen-Macaulay curve Σ of multiplicity three has type ≤ 2, and that $\operatorname{type}(\Sigma) = 1$ implies that Σ is a complete intersection. This happens in case $\lambda(\Sigma) = 0$ (see Lemma 1.4).

LEMMA 1.6. *Let Σ be an isolated curve singularity of multiplicity three and type two, with $\lambda(\Sigma) = \lambda$. Then the tangent cone of Σ is isomorphic to a multiplicity-three scheme $\mathscr{C}$ in $\mathbf{P}^2$ described by*:

Case $\lambda \geq 2$: (y^2, yz, z^2);

Case $\lambda = 1$: Either by (x, z, y^3) or by $(z, y^2) \cap (z, x)$ or by $(z, y) \cap (x, y) \cap (x, z)$.

Proof. This follows easily from the equations describing a multiplicity-three and type-two curve; see (1.4). ⊠

Remark 1.7. As we will see in Section 2, every multiplicity-three and type-two curve with an isolated singularity appears as the reduced singular locus of a projection of a rational quadruple point. It is not true, however, that every curve of type $m - 2$ and multiplicity $(m - 1)(m - 2)/2$ is the singular locus of a projection of a rational singularity of multiplicity m for $m > 5$; there are extra conditions on the curve but we do not know exactly what they are. This is one of the reasons our arguments do not apply for rational singularities of higher multiplicity. Moreover, the work of Arndt [Arn] on cyclic quotient singularities suggests that the results we obtain for rational quadruple points do not have a simple generalization to rational singularities of higher multiplicity.

PROPOSITION 1.8. *Let Σ and coordinates x, y, z be as in Lemma 1.4. Let $I = (\Delta_1, \Delta_2, \Delta_3)$ be the ideal of Σ defined by the minors of the matrix M. Consider the function*

$$\Phi := \det(\tilde{M}) \in \mathbf{C}\{x\}[y, z]; \qquad \tilde{M} := \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ y & z + a & b \\ c & y + d & z \end{pmatrix}$$

where $(\alpha_1, \alpha_2, \alpha_3) := x^{-\lambda} \cdot (dy - cz, ad - bc, az - by)$, $\lambda = \lambda(\Sigma)$. Then Φ has *the following properties*:

(i) $\Phi \in \int I$; i.e., Σ *is in the singular locus of* Φ.

(ii) $\mathrm{Mult}(\Phi) = 3$; $\deg_{(y,z)}(\Phi) = 3$; $\Phi(0, y, z) \neq 0$.

(iii) *Consider a* 3×3 *matrix* h *with entries in* $\mathbf{C}\{x, y, z\}$ *with* generic constant part h_0. *Then the space* X *defined by* $\Phi + \sum h_{ij}\Delta_i\Delta_j = 0$ *has precisely* Σ *as singular locus and has a smooth normalization* $\tilde{X}$, *and the inverse image of* $X \cap \{x = 0\}$ *on* $\tilde{X}$ *is a smooth curve.*

Proof. Let us first indicate the geometrical significance of a function Φ having properties (i) and (ii). The intersection of Σ with the plane $x = t$, $t \neq 0$, consists of three distinct points in the (y, z)-plane. Multiplying together the three linear factors describing the lines through the three pairs of points, we get a polynomial Φ of degree 3 in y and z with coefficients depending on x. A direct computation then shows that Φ can be written as the above determinant. The Cramer matrix $\tilde{N}$ of 2×2 minors of $\tilde{M}$ is seen to be equal to

$$\tilde{N} = \begin{pmatrix} \Delta_1 & \alpha\Delta_2 + \beta\Delta_3 & \alpha\Delta_1 + \beta\Delta_2 \\ \Delta_2 & \gamma\Delta_1 + \delta\Delta_2 & \alpha\Delta_2 + \beta\Delta_3 \\ \Delta_3 & \gamma\Delta_2 + \delta\Delta_3 & \gamma\Delta_1 + \delta\Delta_2 \end{pmatrix}$$

(where $(\alpha, \beta, \gamma, \delta) = -x^{-\lambda} \cdot (a, b, c, d)$), which shows that the matrix $\tilde{N}$ has entries in the ideal I. This is equivalent to the fact that $\Phi \in \int I$ (see [J-S3, (1.12)]), as should be clear geometrically. Now we turn to statement (iii) of the proposition. The curve $X \cap \{x = 0\}$ has an equation of the form $\Phi(0, y, z) + G(y, z)$, where G starts with a *generic* quartic in y and z, because of the genericity of h_0. From this it follows that $\delta(X \cap \{x = 0\}) = 3$. For small values of t the curve $X \cap \{x = t\}$ has three ordinary double points; hence the family $X \cap \{x = t\}$ for varying t is a δ-constant family of plane curves. Consequently, X has precisely Σ as singular locus, $\tilde{X}$ is smooth and the inverse image of $X \cap \{x = 0\}$ on $\tilde{X}$ is smooth (see [L-L-T]).

COROLLARY 1.9. *In the above situation*, $\int I/I^2 \approx \mathbf{C}[x]/x^\lambda$, *and a* $\mathbf{C}$-*basis for* $\int I/I^2$ *is given by* $\Phi, x \cdot \Phi, \ldots, x^{\lambda-1} \cdot \Phi$.

Proof. This can be checked by a direct calculation, but it is much nicer to apply here a beautiful theorem of D. Mond and R. Pellikaan (see [M-P, Thm. 4.4]) which implies that for a weakly normal surface X in $\mathbf{C}^3$ with singular locus Σ and with a Gorenstein normalization $\tilde{X}$, the module $\int I/I^2$ is cyclic with generator F, where $F = 0$ is the equation of X and has as annihilator the $(t-1) \times (t-1)$ minors of the $t \times (t+1)$ matrix defining Σ. In our case $\tilde{X}$ is smooth by Proposition 1.8, (iii) and $t = 2$; so the annihilator of $\int I/I^2$ is the ideal (y, z, x^λ). For a different proof of this fact, see [J-S3, (2.6), (2.8)]. ⊠

2. Weakly normal surfaces with singular locus of multiplicity three and type two

We will prove in this paragraph that weakly normal surfaces, which have a curve of multiplicity three and type two as reduced singular locus, have a base space of a semi-universal admissible deformation isomorphic to a space $B(n) \times \mathbf{C}^k$, where n is a number which can be easily read off from an equation defining the weakly normal surface. As already remarked in the Introduction, we get as a special case the base space of a semi-universal deformation of a rational quadruple point. Finally, we show that for rational quadruple points n can be determined from the resolution graph.

It follows from (1.9) that we have some sort of normal form for equations of weakly normal surfaces X with reduced singular locus Σ a curve of multiplicity three and type two. Every such X is defined by an equation of the form:

$$F_{p,h} := x^p \cdot \Phi + \sum h_{ij}\Delta_i\Delta_j = 0$$

for suitable p and $h = (h_{ij})$.

We will now study how the normalization of a "generic" weakly normal surface with reduced singular locus of multiplicity three and type two looks. We start by studying the tangent cone.

LEMMA 2.1. *Let $X_{p,h}(\Sigma)$ be the surface in $\mathbf{C}^3$ defined by*

$$F_{p,h} = x^p \cdot \Phi + \sum h_{ij}\Delta_i\Delta_j = 0$$

where $h = h_{ij}$ and Δ_i are generators of the ideal of Σ. Suppose h has generic constant part h_0. Then the tangent cone of the surface $X_{p,h}(\Sigma)$ is the cone over a curve $C \subset \mathbf{P}^2$, which has the following structure:

Case A: $\lambda(\Sigma) \geq 2$, $p \geq 2$; *C consists of four distinct lines, all passing through a single common point.*

Case B: $\lambda(\Sigma) \geq 2$, $p = 1$; *C is an irreducible rational quartic curve with a unique singular point of type D_4, D_5 or E_6.*

Case C: $\lambda(\Sigma) = 1$, $p = 1$; *C is an irreducible rational quartic curve with one (A_5), two $(A_3 + A_1)$ or three $(3\ A_1)$ singular points.*

Proof. If $p \geq 2$, then the tangent cone of $X_{p,h}(\Sigma)$ is determined by the term $\sum h_{ij}\Delta_i\Delta_j$, because Φ has multiplicity three. If $\lambda(\Sigma) \geq 2$, then the lowest order terms in the matrix M of (1.4) are the y and z; so for generic h we get as tangent cone a general quartic in y and z, which settles case A. If $p = 1$ and $\lambda(\Sigma) \geq 2$, then the lowest order term of $F_{p,h}$ contains also a term $x \cdot \Phi$. Corresponding to the cases that $\Phi(0, y, z)$ is equivalent to $y^3 + z^3, y^2 \cdot z, y^3$, we then find a D_4, D_5 or an E_6 on C, which settles case B. The remaining case is $\lambda(\Sigma) = 1$. Here we have that the tangent cone of Σ is described (up to isomorphism) by one of the ideals mentioned in Lemma 1.6, and from this it follows easily that the tangent cone $X_{p,h}(\Sigma)$ is as asserted. ⊠

PROPOSITION 2.2. *For generic h and $1 \leq p \leq \lambda(\Sigma)$, surface $X_{p,h}(\Sigma)$ has as normalization a p-star singularity.*

Proof. We blow up $\mathbf{C}^3$ at the origin. Let Σ' and X' be the strict transforms of Σ and $X_{p,h}(\Sigma)$. Now X' will have the tangent cone of $X_{p,h}(\Sigma)$ as exceptional divisor. If $p \geq 2$, then $\lambda(\Sigma) \geq 2$, so that Σ' will still be a curve germ of multiplicity three, and $\lambda(\Sigma') = \lambda(\Sigma) - 1$, as one easily sees from blowing up the matrix M of (1.4). Also, by (2.1), the exceptional divisor of X' consists of four lines through a point, which is also the singular point of Σ'. Around this point the surface X' will have a singularity of type $X_{p-1,h'}(\Sigma')$, as follows if we look at the equation in the x-chart. Because the tangent cone is reduced, X' will be smooth apart from this singularity. As only the constant part of h enters in the genericity assumption for Lemma 2.1, and the constant part of h' is the same as that of h, the same arguments apply for the strict transform of the first blow-up. After $p - 1$ blow-ups we have introduced four chains of rational curves of length $p - 1$ and we are left with a singularity of type $X_{1,h''}(\Sigma'')$. Now there are two cases: $\lambda(\Sigma'') \geq 2$ and $\lambda(\Sigma'') = 1$. These correspond to cases B and C of (2.1). In each of these cases the tangent cone of $X_{1,h''}(\Sigma'')$ is an irreducible rational quartic curve. In the first case we find after still one further blow-up a unique special point of type $X_{0,h'''}(\Sigma''')$, which has by (1.9) a *smooth* normalization (and the inverse image of the quartic is also smooth). In the second case we get, after blowing up $X_{1,h''}(\Sigma'')$, a surface X''' with singular locus Σ''' which can have one, two or three disjoint parts. We claim that X''' again has smooth normalization and that the inverse image of the quartic is also smooth. This can be seen by applying the same idea as in the proof of (1.9): around a part of Σ''' the germ of X''' can be considered as the total space of a family of curves with, as special fibre, the (germ of the) exceptional quartic. It is not hard to see that this is a family with constant δ (equal to 1, 2 or 3), which proves the claim. Our conclusion is that $X_{p,h}(\Sigma)$ for generic h has as normalization a singularity which has, as resolution graph, the graph of the p-star singularity. By keeping track of the order of vanishing of the function x along all exceptional curves, one can compute all the self-intersections and they are as for the p-star singularity. ⊠

Remark 2.3. At this point one can conclude that the base space of a semi-universal admissible deformation of *any* weakly normal surface in $\mathbf{C}^3$ with a curve Σ of multiplicity three and type two as reduced singular locus is (up to a

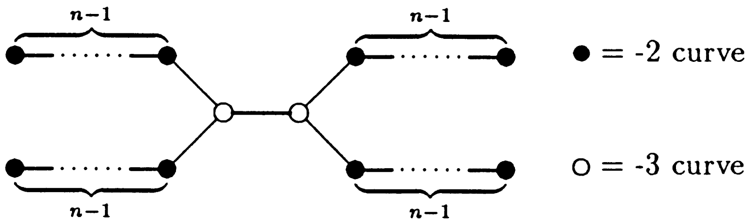

FIGURE 4

smooth factor) isomorphic to the base space of *some* n-star singularity. But, n-star singularities are *not* determined by the analytic type of the resolution graph (cf. [La]), except for $n = 1$ or 2. In fact, one can see from Proposition 3.4 that there is an $n - 1$-dimensional family of n-stars. As a consequence, the argument of the main theorem of this section in an earlier version of this paper [J-S1] is not complete. We are going to find, however, weakly normal surfaces such that the normalization is taut, i.e., determined by the topological data of the resolution graph. We are only able to find such weakly normal surfaces for which a defining function is in I^2. Let $\mathbf{X}(n)$ be the rational quadruple point with dual graph of the minimal resolution (see Figure 4).

THEOREM 2.4 [La]. $\mathbf{X}(n)$ *is taut* ⊠

Construction 2.5. Let Σ be an isolated curve singularity of multiplicity three and type two with $\lambda(\Sigma) = \lambda$, defined by the 2×2 minors of:

$$\begin{pmatrix} y & z + a_\lambda x^\lambda + \cdots & b_\lambda x^\lambda + \cdots \\ c_\lambda x^\lambda + \cdots & y + d_\lambda x^\lambda + \cdots & z \end{pmatrix}.$$

Let Δ_i be the i-th minor of this matrix. Consider the (maybe nonreduced) curve singularity defined by the 2×2 minors of

$$\begin{pmatrix} y & z + a_\lambda x & b_\lambda x \\ c_\lambda x & y + d_\lambda x & z \end{pmatrix}$$

and let $\overline{\Delta}_i$ be the i-th minor of this matrix. Because this ideal is homogeneous, this defines a multiplicity-three scheme $\mathscr{C}$ in $\mathbf{P}^2$. By Lemma 1.6 we know the possible isomorphism classes of $\mathscr{C}$, and it is therefore easy to construct smooth quadrics Q_1: $\overline{\Phi}_1 = 0$, Q_2: $\overline{\Phi}_2 = 0$ such that:

(1) $\mathscr{C}$ is contained in Q_1 and Q_2, Q_1 is not equal to Q_2.

(2) The fourth point of intersection of Q_1 and Q_2 is not contained in the reduction of $\mathscr{C}$.

(3) $\overline{\Phi}_1(x, y, z) \cdot \overline{\Phi}_2(x, y, z) = 0$ intersects the line $x = 0$ in four different points.

Now write $\overline{\Phi}_1 = \Sigma p_i \overline{\Delta}_i$; $\overline{\Phi}_2 = \Sigma q_i \overline{\Delta}_i$, and put

$$\Phi_1 = \sum p_i \Delta_i \quad \text{and} \quad \Phi_2 = \sum q_i \Delta_i.$$

Let h_{ij} be a generic 3×3 matrix with entries complex numbers. Let $X(\Sigma)$ be the singularity defined by the equation:

$$\Phi_1 \cdot \Phi_2 + x \cdot \sum h_{ij} \Delta_i \Delta_j = 0.$$

PROPOSITION 2.6. *The normalization of* $X(\Sigma)$ *is isomorphic to* $\mathbf{X}(\lambda)$.

Proof. This is completely analogous to the proof of Proposition 2.2. The only difference is that after $\lambda - 1$ blow-ups, one does not get an irreducible rational quartic, but, by construction, two quadrics. Details are left to the reader. ⊠

Let us briefly recall the description of the normalization mapping that was used in [J-S3]. Let I be an ideal in $\mathscr{O}_X$ that satisfies the so-called *ring condition* $\operatorname{Hom}_X(I, I) = \operatorname{Hom}_X(I, \mathscr{O}_X)$. Then we get a ring extension $\mathscr{O}_X \subset \mathscr{O}_{\tilde{X}} := \operatorname{Hom}_X(I, I)(\subset Q(\mathscr{O}_X))$, where we put $\tilde{X} := \operatorname{Spec}(\operatorname{Hom}_X(I, I))$. Conversely, given $\mathscr{O}_X \subset \mathscr{O}_{\tilde{X}}$, we can reconstruct I as the *conductor ideal* $\operatorname{Hom}_X(\mathscr{O}_{\tilde{X}}, \mathscr{O}_X)$. In particular, this applies to the normalization $\tilde{X} \to X$ of a weakly normal surface X with singular locus described by an ideal I. It is important to note that this construction of $\mathscr{O}_{\tilde{X}}$ works perfectly well even if X is not reduced. We apply this construction to the non-reduced space defined by the equation $x^p \cdot \Phi = 0$.

LEMMA 2.7. *Let I be generated by the* 2×2 *minors of M of* (1.4) *and let Y be defined by the equation* $x^p \cdot \Phi = 0$. *Then the space* $\tilde{Y} \subset \mathbf{C}^5$ *is defined by the following six equations*:

$$\begin{aligned}
x^p \cdot \alpha_1 + u \cdot y + v \cdot c &= 0, & u^2 &= x^p \cdot (\delta u + \gamma v), \\
x^p \cdot \alpha_2 + u \cdot (z + a) + v \cdot (y + d) &= 0, & uv &= x^p \cdot (\alpha u + \delta v) \\
& & &\quad + x^{2p} \cdot (\beta\gamma - \alpha\delta), \\
x^p \cdot \alpha_3 + u \cdot b + v \cdot z &= 0, & v^2 &= x^p \cdot (\beta u + \alpha v)
\end{aligned}$$

with notation as in (*the proof of*) *Proposition* 1.8.

Proof. As a module over $\mathscr{O}_Y$, $\mathscr{O}_{\tilde{Y}}$ is isomorphic to the cokernel of the matrix $\tilde{M}$ of (1.8), with the top row multiplied by x^p. The rows of this matrix correspond to the elements 1, u and v of $\mathscr{O}_{\tilde{Y}}$; hence the first three equations hold. To get the ring structure on $\mathscr{O}_{\tilde{Y}}$, we have to compute the products u^2, uv and v^2. But the columns of the Cramer matrix $\tilde{N}$ correspond to 1, u and v, respectively, which are seen as elements of $\operatorname{Hom}_Y(I, I)$ and with the explicit form of $\tilde{N}$ given in (1.8), we find the other three equations. ⊠

PROPOSITION 2.8. *Let Σ be an isolated curve singularity of multiplicity three and type two, with $\lambda(\Sigma) = \lambda$. Let $[f]$ be an element of $\int I/I^2$. Then there exists a representative f of $[f]$ such that*:

(1) *The singular locus of f is exactly Σ, and the surface X defined by $f = 0$ is weakly normal.*

(2) *The normalization X of X has a projection $\tilde{X} \to X'$, with defining equation $f' = 0$ for X' and defining ideal I' for the reduced singular locus Σ' of X' such that $f' \in I'^2$. Furthermore, Σ' is a curve of multiplicity three and type two.*

Proof. One can assume that the ideal I is generated by the 2×2 minors of the matrix in (1.4) and that $[f] = [x^p\Phi]$. Take the lift $x^p\Phi$ of f and let the space Y be defined by this lift. The equations of the space $\tilde{Y} = \operatorname{Spec}(\operatorname{Hom}_Y(I, I))$ are given in Lemma 2.7. Consider the following change of coordinates:

$$x' = x,\ v' = v,\ u' = u,\ z' = z - v,\ y' = y - u.$$

Performing the substitution in the first three equations of (2.7) and using the second three equations of (2.7), we then see that $\mathcal{O}_{\tilde{Y}}$, considered as a module over $\mathbf{C}\{x', y', z'\}$, is equal to the cokernel of the matrix:

$$\tilde{M}' := \begin{pmatrix} \alpha'_1 & \alpha'_2 & \alpha'_3 \\ y' & z' + a' & b' \\ c' & y' + d' & z' \end{pmatrix}$$

where $a' = a + 2 \cdot x^p\alpha$, $b' = b + 2 \cdot x^p\beta$, $c' = c + 2 \cdot x^p\gamma$, $d' = d + 2 \cdot x^p\delta$, $\alpha'_1 = \alpha_1$, $\alpha'_2 = \alpha_2 + 2 \cdot x^{2p} \cdot (\beta\gamma - \alpha\delta)$ and $\alpha'_3 = \alpha_3$.

Let $I'' = (\Delta'_1, \Delta'_2, \Delta'_3)$ be the ideal generated by the 2×2 minors of (the lower part of) the matrix $\tilde{M}'$. One calculates that $(x^{\lambda-p} - 2) \cdot \det(\tilde{M}') = \Delta'_1 \cdot \Delta'_3 - (\Delta'_2)^2$. The image Y' under the projection $(x', y', z', u', v') \to (x', y', z')$ is given by the equation $\det(\tilde{M}') = 0$ and the role of the curve Σ is replaced by the curve Σ'', defined by the ideal I''. In particular, we see that $\lambda(\Sigma'') = p$.

Now consider a function $g \in I^2$, such that $f = x^p\Phi + t \cdot g$, t a small parameter, has exactly Σ as singular locus, and $f = 0$ is weakly normal. That such a function g exists can be proved in exactly the same way as in [Pe (2.1)]. Let $\tilde{Z}$ be the normalization of Z, defined by $f = 0$. By [J-S3, (1.3)], $\tilde{Z}$ can be considered as the total space of a deformation of $\tilde{Y}$. Here $\tilde{Z} \subset \mathbf{C}^5 \times T$. Taking the projection $\mathbf{C}^5 \times T \to \mathbf{C}^3 \times T$ described by $(x', y', z', u', v', t) \to (x', y', z', t)$, we get a small admissible deformation of Y', again by [J-S3, (1.3)]. Call a general fibre of this deformation X'. The singular locus Σ' of X' has λ invariant less than or equal to p, because it is a small deformation of the curve Σ''. Now the invariant $\dim \int I/(I^2 + f)$ only depends on the normalization of

$f = 0$ [J-S3, (2.6)]; so one deduces that the λ invariant of Σ' is exactly p and that $f' \in I'^2$, where f' defines X', and I' defines Σ'. ⊠

Remark 2.9. We expect that the normalization of every weakly normal surface, which has as reduced singular locus a multiplicity-three and type-two curve, has a projection X' into $\mathbf{C}^3$, such that for a defining function f' of X' one has $f' \in I'^2$, I' the ideal of the reduced singular locus Σ' of X'. We do not know of a proof, except maybe by a very tedious calculation. We remark, moreover, that this is a very peculiar property of multiplicity-three and type-two curves and is certainly not true for most curves in $\mathbf{C}^3$.

We are now in the position to prove the main theorem of this section:

THEOREM 2.10. *Let X be a weakly normal surface in $\mathbf{C}^3$ with reduced singular locus Σ of multiplicity three and type two. Let I be the ideal defining Σ and $f = 0$ be an equation for X. Then the base space of a semi-universal admissible deformation of X is isomorphic to a space $B(n) \times \mathbf{C}^k$, where $n = \dim(\int I/(I^2 + f))$. Moreover, the same is true for the base space of a semi-universal deformation of a rational quadruple point.*

Proof. Let us recall from [J-S2] that two functions $f, g \in \int I$ are called I^2 equivalent if and only if $\int I/(I^2 + f) = \int I/(I^2 + g)$. From [J-S2, (1.16)] one has that if two functions are I^2-equivalent, then the base spaces of semi-universal admissible deformations of f and g are isomorphic up to a smooth factor. Also from [J-S3, (1.4)], base spaces of semi-universal deformations of two weakly normal surfaces are isomorphic up to a smooth factor if they have isomorphic normalizations. Using these two facts and Proposition 2.8, one reduces to the case that a defining function of X is in I^2. Proposition 2.6 tells us that every curve Σ of multiplicity three and type two and $\lambda(\Sigma) = n$ has a function in I^2 such that the normalization is isomorphic to the rational quadruple point $\mathbf{X}(n)$. From this the theorem follows. The statement about rational quadruple points follows from the fact that a generic projection of a rational quadruple point has as reduced singular locus a curve of multiplicity three and type two by (1.3). ⊠

For a rational quadruple point it is possible to determine $\dim(\int I/(I^2 + f))$ (where $f = 0$ is a defining equation for a generic projection) from the resolution graph. For this we need the following general lemma.

LEMMA 2.11. *Let $\tilde{X}$ be a normal surface singularity $\subset \mathbf{C}^N$ and let X be the image of $\tilde{X}$ under a linear projection L: $\mathbf{C}^N \to \mathbf{C}^3$. Let $\tilde{X}_1$ and X_1 be the blow-ups* *of $\tilde{X}$ and X, respectively. Then for a Zariski-open set of L's:*

(1) *There is an induced map $\tilde{X}_1 \xrightarrow{L_1} X_1$.*

(2) *The normalization of $\tilde{X}_1$ is isomorphic to the normalization of X_1.*

Proof. For a vector space V, let $\mathbf{P}(V)$ be the projective space of lines through 0 and let V_1 be the blow-up of V at 0 (or the tautological bundle over $\mathbf{P}(V)$): $V_1 = \{(x, \ell) \in V \times \mathbf{P}(V) | x \in \ell\}$. Let $L: V \to W$ be a linear surjection with kernel K. The inclusion $K \subset V$ induces an inclusion $K_1 \subset V_1$ and L induces a map L_1: $U_1 := V_1 - K_1 \to W_1$, exhibiting U_1 as a rank equal to $\dim(K)$-vector bundle. Let $\tilde{X}$ be a germ in V and $X := L(\tilde{X}) \subset W$. Let $\tilde{X}_1$ and X_1 be the strict transforms of $\tilde{X}$ and X, respectively. The tangent cone $C(\tilde{X})$ is just $\tilde{X}_1 \cap \mathbf{P}(V)$. Now if $C(\tilde{X}) \cap \mathbf{P}(K) = \emptyset$, one also has $\tilde{X}_1 \cap K_1 = \emptyset$, so that L_1 induces a map $\tilde{X}_1 \to X_1$, mapping $C(\tilde{X})$ to $C(X)$. In the case that $C(\tilde{X})$ is mapped generically one-to-one to $C(X)$, the same is true for $\tilde{X}_1 \to X$. Thus, under these circumstances, $\tilde{X}_1$ and X_1 will have the same normalization. A simple dimension count involving the secant variety of $C(\tilde{X})$ then shows that these conditions are satisfied for a Zariski-open set of L's as soon as $\dim(\tilde{X}) \leq \dim W - 1$. In particular this applies to projections of surface germs to $\mathbf{C}^3$. ⊠

Definition 2.12. Let $\tilde{X}$ be a rational quadruple point. Then $n(\tilde{X})$ is defined inductively by:

(1) If on the strict transform $\tilde{X}_1$ of $\tilde{X}$ of the first blow-up no rational quadruple point occurs, then $n(\tilde{X}) := 1$.

(2) If on $\tilde{X}_1$ there is a rational quadruple point at, say, p, then $n(\tilde{X}) := n(\tilde{X}_1, p) + 1$.

By results of Tjurina [Tj], the strict transform of the first blow-up of a rational singularity is normal and the singularities appearing on the blow-up are easy to describe in terms of the resolution graph of the original singularity. Using this, we easily calculate $n(\tilde{X})$ from the resolution graph of $\tilde{X}$.

THEOREM 2.13. *Let $\tilde{X} \to X \subset \mathbf{C}^3$ be a generic projection of a rational quadruple point. Let Σ be the reduced singular locus of X and let I, f define Σ, X, respectively. Then*

$$n(\tilde{X}) = \dim\left(\int I/(I^2 + f)\right).$$

Proof. Because the projection is generic (in the sense that (1.2) applies), the singular locus of X is a curve Σ of multiplicity three and type two, with $\lambda(\Sigma) = \lambda$. So, by (1.9), one can write $f = x^p\Phi + \Sigma h_{ij}\Delta_i\Delta_j$ for a certain matrix h_{ij}, with $p \leq \lambda$. Suppose first that $p = \lambda \geq 2$. Then on the first blow-up X_1 of X one has a special non-isolated singularity where the singular locus Σ_1 has

$\lambda(\Sigma_1) = \lambda - 1$, the normalization of which occurs on the first blow-up $\tilde{X}_1$ of $\tilde{X}$, by Lemma 2.11. This must be a rational quadruple point, because it deforms into one. (Perturb the matrix h_{ij} until it becomes generic; then the normalization is a $(\lambda - 1)$-star singularity; see (2.2).) Because $\tilde{X}$ is resolved by a finite number of blow-ups in points, we may assume that for each of the local singularities of $\tilde{X}_1$ the genericity conditions of (2.11) hold for the map $L_1\colon \tilde{X}_1 \to X_1$ as well (in local charts L_1 is linear). So, by induction, we reduce to the case $\lambda = 1$. If one blows up once more, then the strict transform of the singular locus becomes a curve of type one (hence a complete intersection). The normalization of the strict transform of X now has embedding dimension less than or equal to 4, hence cannot be a rational quadruple point. This proves the theorem in the case $p = \lambda$. The case $p \leq \lambda - 1$ is similar, and therefore left to the reader. ⊠

Remark 2.14. It is proved in [J-S3, (2.8)] that $\dim(\textstyle\int I/(I^2 + f) = \dim \operatorname{Ext}^1(\omega_{\tilde{X}}, \mathcal{O}_{\tilde{X}})$ for any projection of $\tilde{X}$. Hence it follows from Theorem 2.13 that $n(\tilde{X}) = \operatorname{Ext}^1(\omega_{\tilde{X}}, \mathcal{O}_{\tilde{X}})$ for rational quadruple points. For another proof and another interpretation of this number, see the recent paper of J. Stevens [St, Lemma 8].

3. The semi-universal deformation of an n-star

By the results of Section 2, in particular Theorem 2.10, the base space of a semi-universal deformation of an arbitrary rational quadruple point $\tilde{X}$ will be, up to smooth factors, equal to that of any $n(\tilde{X})$-star singularity. In this paragraph we determine a semi-universal deformation of a particular n-star singularity that has a very symmetrical projection into $\mathbf{C}^3$. We do this essentially by computing the semi-universal admissible deformation of the projection. We include the calculation in some detail as it is a good illustration of our theory of admissible deformations (the remaining details can be filled in easily by the reader). Although the main result of this section is straightforward, one should not underestimate the effort to prove such a result. The main difficulty is finding the right notation and making the right choices. The symmetry that runs through all calculations is of great help, but formalizing this (in terms of representations) did not increase our understanding of this mysterious calculation.

We begin with the study of a very special curve Σ of multiplicity three and type two and λ-invariant (see (1.4)), equal to a fixed natural number n greater than zero.

Definition 3.1. Let $L_i(y, z) \in \mathbf{C}[y, z]$ $(i = 1, 2, 3)$ be three different linear forms with $L_1 + L_2 + L_3 = 0$ and put $M_i = L_i + x^n \in \mathbf{C}[x, y, z]$. The curve

$\Sigma \subset \mathbf{C}^3$ is the curve singularity defined by the ideal

$$I = (\Delta_1, \Delta_2, \Delta_3) = (M_2M_3, M_3M_1, M_1M_2).$$

So Σ consists of three smooth branches and each pair of branches has contact order equal to n. Each M_i describes a "bended plane" through two of the three branches. There is an obvious action of S_3 on Σ obtained by permuting the branches.

PROPOSITION */Definition* 3.2.

(a) *The normal sheaf* $N_\Sigma := \mathrm{Hom}_\Sigma(I, \mathscr{O}_\Sigma)$ *of* Σ *is generated as an* $\mathscr{O}_\Sigma$-*module by* $n_1, n_2, \ldots, n_6$ *whose values on* $\Delta_1, \Delta_2, \Delta_3$ *are*:

	n_1	n_2	n_3	n_4	n_5	n_6
Δ_1	M_3	M_2	0	0	0	0
Δ_2	0	0	M_1	M_3	0	0
Δ_3	0	0	0	0	M_2	M_1

(b) *A basis for* T^1_Σ *is given by the classes of the normal vectors*

$$x^qA_i \quad (q = 0, \ldots, n-1;\ i = 1,2,3) \text{ and}$$

$$x^qB \quad (q = 0, \ldots, n-2).$$

Here

$$A_1 = -n_1 + n_2, \qquad A_2 = -n_3 + n_4, \qquad A_3 = -n_5 + n_6;$$

$$B = n_1 + \cdots + n_6.$$

(c) *Let* $a_i := \sum_{j=0}^{n-1} a_{ij}x^j$ $(i = 1,2,3)$ *and let* $b := \sum_{j=0}^{n-2} b_jx^j$.

Let $S := \mathbf{C}[a_1, a_2, a_3, b]$ *be the polynomial ring in the coefficients* a_{ij} *and* b_j *of the polynomials* a_i *and* b. *Then a semi-universal deformation of* Σ_B *of* Σ *over* $B = \mathrm{Spec}(S)$ *is described by the ideal*

$$I_B = (\Xi_1, \Xi_2, \Xi_3) \in S[x, y, z]$$

where, with $N_i := M_i + b$:

$$\Xi_1 = N_2N_3 + a_1(N_2 - N_3) + a_1a_2 + a_1a_3 + a_2a_3.$$

$$\Xi_2 = N_1N_3 + a_2(N_3 - N_1) + a_1a_2 + a_1a_3 + a_2a_3.$$

$$\Xi_3 = N_1N_2 + a_3(N_1 - N_2) + a_1a_2 + a_1a_3 + a_2a_3.$$

(d) *The normal sheaf* $N_{\Sigma_B} := \mathrm{Hom}(I_B, \mathcal{O}_{\Sigma_B})$ *of* Σ_B *is generated as an* $\mathcal{O}_{\Sigma_B}$*-module by* $\nu_1, \nu_2, \ldots, \nu_6$ *whose values on* Ξ_1, Ξ_2, Ξ_3 *are*:

	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6
Ξ_1	$N_3 - a_2$	$N_2 + a_3$	$-a_1 - a_3$	0	0	$a_1 + a_2$
Ξ_2	0	$a_2 + a_3$	$N_1 - a_3$	$N_3 + a_1$	$-a_2 - a_1$	0
Ξ_3	$-a_3 - a_2$	0	0	$a_3 + a_1$	$N_2 - a_1$	$N_1 + a_2$

We leave the straightforward proof to the reader and note that the deformation with $a_i = 0$ for general values of b will create a curve having n triple points. The result (d) will be needed later on.

Definition 3.3. Let $f := \Delta_1^2 + \Delta_2^2 + \Delta_3^2$ and let X be defined by the equation $f = 0$.

Now X is a weakly normal surface singularity in $\mathbf{C}^3$ having exactly the curve Σ of (3.1) as singular locus. (A different real form of) X looks something like Figure 5.

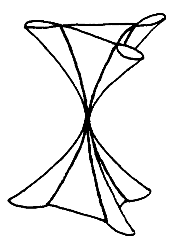

FIGURE 5

PROPOSITION 3.4.
(a) *The normalization* $\tilde{X}$ *of* X *is an* n*-star singularity.*
(b) $\dim T^1_{\tilde{X}} = 6n - 2$. *The following elements of* $T^1(\Sigma, X) \approx T^1(\tilde{X} \to X)$ *project onto a basis of* $T^1_{\tilde{X}}$:

$$\begin{array}{lll} \mathrm{I} & x^q M_1 M_2 M_3, & q = 0, \ldots, n-1; \\ \mathrm{II} & 2x^q A_i \cdot \Delta, & q = 0, \ldots, n-1, \quad (i = 1,2,3); \\ \mathrm{III} & 2x^q B \cdot \Delta, & q = 0, \ldots, n-2; \\ \mathrm{IV} & x^q \Delta_2 \Delta_3, & q = 0, \ldots, n-2. \end{array}$$

(c) $c_{I,e}(f) = \dim \int I / (\int I \cap J(f)) = 9n - 5$, $\quad \dim T^1(\Sigma, X) = 13n - 6$,
$j(f) = \dim I/J(f) = 16n - 6$, $\quad VD_\infty(f) = 4n$.

Proof. Statement (a) follows as in the proof of Proposition 2.2. For (b) one has to rely on [J-S3, Th. 3.1], but let us point out the geometrical significance of the indicated deformations. By Corollary 1.9, $\int I/I^2$ is a cyclic module generated by Φ. In this case $\Phi = M_1M_2M_3$ and the first n infinitesimal admissible deformations of type I are obtained by adding $\varepsilon \cdot x^q\Phi$ to the defining function f of X. Furthermore, because $f \in I^2$ we get admissible deformations by replacing the Δ_i by Ξ_i in f. In first order, this gives the deformations II and III, corresponding to the A and B deformations of Σ in Proposition 3.2. (Here we used the shorthand notation $B \cdot \Delta = \sum_{i=1}^{3} B(\Delta_i) \cdot \Delta_i$, etc.) The deformations of type IV are deformations that do not deform Σ and keep f in I^2. In this case, these correspond to the *moduli* of $\tilde{X}$, and it is hoped the reader will realize that these are unimportant for our purposes. Statement (c) lists the values of some important invariants of X that figure in [J-S2] and [J-S3]. These are included for completeness only and are not used in the sequel. ⊠

Definition 3.5. Let $e := \sum_{j=0}^{n-1} e_j \cdot x^j$ and let $R := S[\underline{e}] = \mathbf{C}[\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{b}, \underline{e}]$ be the polynomial over the ring S of Proposition 3.2(c) in the coefficients of the polynomial e. Now in the ring $R[x]$ consider the algorithm of division by $x^n + b$ with remainder; i.e., we write

$$e \cdot a_i = (x^n + b)[ea_i] + E_i \qquad (i = 1,2,3).$$

with the degree of E_i in x less than n. Hence $[e \cdot a_i]$ and E_i are polynomials in x with certain universal elements of R as coefficients. Let $J(n)$ be the ideal in R generated by the coefficients of the polynomials E_1, E_2, E_3 and let

$$B(n) := \operatorname{Spec}(R/J(n)).$$

Furthermore, let $\mathbf{m} = (\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{b}, \underline{e})$ be "the" maximal ideal of the ring R.

In the course of proving the main result of this paragraph, Theorem 3.8, we will need the following lemma.

Lemma 3.6. *The following relations hold*:

$$E_i \cdot a_j \equiv E_j \cdot a_i \qquad \text{modulo}(x^n + b)\mathbf{m}J(n)$$

$$[ea_i] \cdot a_j \equiv [ea_j] \cdot a_i \qquad \text{modulo } \mathbf{m}J(n).$$

Proof. We do division with remainder in $R[x]$:

$$E_i a_j = (x^n + b)[E_i \cdot a_j] + R_{ij}, \text{ with degree of } R_{ij} \text{ in } x \text{ less than } n.$$

Then

$$\begin{aligned}(e \cdot a_i) \cdot a_j &= \left\{(x^n + b)[ea_i] + E_i\right\} \cdot a_j \\ &= (x^n + b)\left\{[ea_i] \cdot a_j + [E_i a_j]\right\} + R_{ij}.\end{aligned}$$

Changing the roles of i and j, we see that $R_{ij} = R_{ji}$, and hence

$$E_i \cdot a_j - E_j \cdot a_i = (x^n + b)\{[E_i a_j] - [E_j a_i]\} \in (x^n + b)\mathbf{m}J(n).$$

The second assertion is also easy. ⊠

Notation/Convention 3.7. Let

$$\begin{aligned} V_1 &= (0, 1, -1), & W_1 &= (0, -N_3, N_2),\\ V_2 &= (-1, 0, 1), & W_2 &= (N_3, 0, -N_1),\\ V_3 &= (1, -1, 0), & W_3 &= (-N_2, N_1, 0), \end{aligned}$$

and let the vector α' be defined by

$$\alpha' = e(N_1, N_2, N_3) + \sum_i e \cdot a_i V_i + \sum_j [ea_i] W_i + \sum_{i<j} [ea_i a_j](1, 1, 1).$$

Furthermore, we will adopt the "inproduct convention" by writing $X \cdot Y := \Sigma X_i \cdot Y_i$ for any two symbols X and Y indexed by the same index set.

THEOREM 3.8. *The base space of a semi-universal deformation of the n-star $\tilde{X}$ is isomorphic to the space $T = B(n) \times \mathbf{C}^{n-1}$. A projection of the total space $\tilde{X}_T$ of a semi-universal deformation of $\tilde{X}$ is the hypersurface X_T described by the equation*:

$$\alpha \cdot \Xi = 0.$$

Here

$$\begin{aligned} \alpha &= \Xi + \alpha' + \left(\sum_{j=0}^{n-2} c_j \cdot x^j\right)(0, 0, \Xi_2),\\ \Xi &= (\Xi_1, \Xi_2, \Xi_3). \end{aligned}$$

Proof. Because the functor $\mathrm{Def}(\tilde{X} \to X)$ is naturally equivalent to the functor $\mathrm{Def}(\Sigma, X)$ of admissible deformations, and the natural forgetful functor $\mathrm{Def}(\tilde{X} \to X) \to \mathrm{Def}(\tilde{X})$ is smooth ([J-S3]), every admissible deformation of X induces a deformation of $\tilde{X}$ and every deformation of $\tilde{X}$ can be lifted to an admissible deformation of X. Hence, by Schlessinger's construction of a (formal) semi-universal deformation ([Sch, 2.11]), we have to find ideals

$$J_k \subset U := = \mathbf{C}[[\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{b}, \underline{e}, \underline{c}]] \text{ and deformations}$$
$$\xi_k \in \mathrm{Def}(\Sigma, X)(U/J_k)$$

such that:

$*$ $J_2 = \mathbf{m}^2$ and ξ_2 induces a semi-universal deformation of $\tilde{X}$ to first order.

$**$ J_{k+1} is an ideal such that $J_k \supset J_{k+1} \supset \mathbf{m} \cdot J_k$ and such that J_{k+1} is minimal with respect to the property that ξ_k lifts to ξ_{k+1}.

Proposition 3.4 (b) means just that $*$ holds. As for $**$, we are going to prove that the ideals $J_k := J(n) + \mathbf{m}^k$ (with $J(n)$ as in Definition 3.5) and $\xi_k = \xi$

modulo J_k (with ξ the family as in the statement of the theorem) will do the job. In general, a way to construct a minimal J_{k+1} out of J_k and ξ_k is by obstruction theory. Let us recall how this works for the functor of admissible deformations. For details we refer to [J-S2]. Consider a small extension of the form

$$0 \to J/\mathbf{m}J \to U/\mathbf{m}J \to U/J \to 0$$

and let $\xi \in \mathrm{Def}(\Sigma, X)(U/J)$. Suppose that ξ is described by

$$f_\xi = \alpha \cdot \Xi := \Sigma \alpha_i \cdot \Xi_i$$

where $f_\xi = 0$ is an equation for X_ξ and Ξ_i generators of the ideal of Σ_ξ. By [J-S2], admissibility of the family means that one can find, for every normal vector $\nu \in \mathrm{Hom}(\Xi, \mathscr{O}_{\Sigma_\xi})$, a $\gamma = \gamma_\nu$ such that the following relation holds:

$$\alpha \cdot \nu + \gamma_\nu \cdot \Xi (:= \Sigma \alpha_i \cdot \nu(\Xi_i) + \Sigma \gamma_i \cdot \Xi_i) = 0 \bmod J.$$

The obstruction element is defined as follows.

$$\mathrm{Ob}(\xi) = \{n \mapsto \alpha \cdot \nu + \gamma_\nu \cdot \Xi\} \in J/\mathbf{m}J \otimes N^*/I.$$

Here ν is any lift of $n \in N = \mathrm{Hom}_\Sigma(I, \mathscr{O}_\Sigma)$ and $N^* = \mathrm{Hom}_\Sigma(N, \mathscr{O}_\Sigma)$. In particular, this applies to $J = J_k$ and $\xi = \xi_k$, giving us an obstruction element $\mathrm{Ob}(\xi_k)$. Then a minimal ideal J_{k+1} is constructed from J_k and ξ_k by $J_{k+1} := (V_1, \dots, V_\sigma) + \mathbf{m}J_k$, where $V_i := v_i(\mathrm{Ob}(\xi_k))$, v_i a dual basis of N^*/I.

So, in order to show that the system of ideals $J_k = J(n) + \mathbf{m}^k$ satisfies condition $**$, we must have:

(1) ξ is an admissible family; i.e., for every ν of Proposition 3.2 (d) there exists a γ_ν such that $\alpha \cdot \nu + \gamma_\nu \cdot \Xi = 0 \bmod J(n)$.

(2) $(v_1(\mathrm{Ob}(\xi)), \dots, v_\sigma(\mathrm{Ob}(\xi))) = J(n)/\mathbf{m}J(n)$

because then, by induction, $J_k = J(n) + \mathbf{m}^k$.

For a function $\alpha'' \cdot \Xi$ in $(\Xi)^2$ it is easy to find for each ν a γ_ν such that $\alpha'' \cdot \nu + \gamma_\nu \cdot \Xi = 0$, not only modulo $J(n)$. (This is the idea of I^2-equivalence; see [J-S2].) So we may as well first replace α by the α' of (3.7) and then look for the appropriate γ's. In Definition 3.9 we define certain γ's and Lemma 3.10 contains a proof that these γ's have the above properties.

Definition 3.9. Six vectors, γ_i, $i = 1, 2, \dots, 6$, are defined as follows:

$$\begin{aligned}
\gamma_1 &= -(0,e,0) + ([ea_2]+[ea_3],\ [ea_2],\ 0)\\
\gamma_2 &= -(0,0,e) - ([ea_2]+[ea_3],\ 0,\ [ea_3])\\
\gamma_3 &= -(0,0,e) + (0,\ [ea_1]+[ea_3],\ [ea_3])\\
\gamma_4 &= -(e,0,0) - ([ea_1],\ [ea_1]+[ea_3],\ 0)\\
\gamma_5 &= -(e,0,0) + ([ea_1],\ 0,\ [ea_1]+[ea_2])\\
\gamma_6 &= -(0,e,0) - (0,\ [ea_2],\ [ea_1]+[ea_2]).
\end{aligned}$$

LEMMA 3.10. (a) *The dual N^* of the normal sheaf N_Σ is generated as an $\mathcal{O}_\Sigma$-module by the six elements n_i^* $(i = 1, 2, \ldots, 6)$, of which the only non-zero values on the six normal vectors n_i of* 3.2 (a) *are*:

$$n_3^*(n_3) = n_6^*(n_6) = M_1,$$
$$n_2^*(n_2) = n_5^*(n_5) = M_2,$$
$$n_1^*(n_1) = n_4^*(n_4) = M_3.$$

(b) *The dimension of the obstruction space N^*/I is $3n$, and a basis is given by $x^q(n_4^* - n_5^*)$, $x^q(n_6^* - n_1^*)$, $x^q(n_2^* - n_3^*)$, $q = 0, 1, 2, \ldots, n-1$.*

(c) *With all the notation introduced above, the following six equations hold modulo $\mathbf{m}J(n)$*:

$$\alpha'\nu_1 + \gamma_1\Xi = -3E_2M_3;\ \alpha'\nu_3 + \gamma_3\Xi = -3E_3M_1;\ \alpha'\nu_5 + \gamma_5\Xi = -3E_1M_2,$$
$$\alpha'\nu_2 + \gamma_2\Xi = 3E_3M_2;\ \alpha'\nu_4 + \gamma_4\Xi = 3E_1M_3;\ \alpha'\nu_6 + \gamma_6\Xi = 3E_2M_1.$$

(d) *The obstruction element of the family of Theorem* 3.8 *in $J(n)/\mathbf{m}J(n) \otimes N^*/I$ is equal to $E_1(n_4^* - n_5^*) + E_2(n_6^* - n_1^*) + E_3(n_2^* - n_3^*)$.*

Proof. Statements (a) and (b) are proved by a straightforward calculation. The identities (c) really come to the heart of the matter. Because we can use symmetry, we have to check only the first identity. Writing out $\alpha'\nu_1 + \gamma_1\Xi$ we find:

$$\begin{aligned}\alpha'\nu_1 + \gamma_1 \cdot \Xi = {} & -3ea_2N_3 - 3ea_2a_3 + (N_1 + N_2 + N_3)[ea_2]N_3 \\ & + \left(\sum [ea_ia_j] - [ea_2]a_1 - [ea_3]a_1\right) \cdot N_3 \\ & + ([ea_3]a_2 + [ea_2]a_1 - [ea_1]a_2 + [ea_3]a_1 - [ea_1]a_3) \cdot N_2 \\ & + [ea_2]a_3N_1 \\ & + \left(\sum a_ia_j\right)(2[ea_2] + [ea_3]) - \left(\sum [ea_ia_j]\right)(2a_2 + a_3).\end{aligned}$$

By Lemma 3.6 this reduces modulo $\mathbf{m}J(n)$ to:

$$\begin{aligned}\alpha' \cdot \nu_1 + \gamma_1 \cdot \Xi = {} & -3ea_2N_3 - 3ea_2a_3 + (N_1 + N_2 + N_3)[ea_2]N_3 \\ & + [ea_2a_3](N_1 + N_2 + N_3).\end{aligned}$$

Now remember that

$$N_i = M_i + b,$$
$$N_1 + N_2 + N_3 = 3(x^n + b),$$
$$(x^n + b)[ea_2] = ea_2 - E_2,$$
$$(x^n + b)[ea_2a_3] = ea_2a_3 - E_2a_3.$$

Thus we find: $\alpha' \cdot \nu_1 + \gamma_1 \cdot \Xi = -3E_2N_3 \equiv -3E_2M_3$ modulo $\mathbf{m}J(n)$. Statement (d) follows directly from (c). ⊠

4. The structure of the base space

The space $B(n)$ of Definition 3.5 is defined by conditions that are conceptually very simple: take four polynomials a_1, a_2, a_3 and e of degree $n-1$ with indeterminates as coefficients and a similar polynomial b of degree $n-2$. Then $B(n)$ is defined by the condition that ea_i be divisible by $x^n + b$ for $i = 1, 2, 3$. To give the reader an idea about how these equations look, we write them out for $n = 1$ and $n = 2$.

Example 4.1.

$n = 1$: $ea_1 = ea_2 = ea_3 = 0$.

$n = 2$: $e_0a_{10} - be_1a_{11} = e_0a_{20} - be_1a_{21} = e_0a_{30} - be_1a_{31} = 0$ and

$$e_0a_{11} + e_1a_{10} = e_0a_{21} + e_1a_{20} = e_0a_{31} + e_1a_{30} = 0.$$

Of course, the case $n = 1$ is the base space of the Pinkham example [Pi]: a one-dimensional linear space transverse to a three-dimensional linear space. For $n = 2$, the equations are already a bit harder to analyse. The space has components of dimension 3, 5 and 7, the five-dimensional component being singular; it is not even Cohen-Macaulay. The primary decomposition of the ideal is:

$$J(2) = (e_0, e_1)$$
$$\cap\left(e_0^2 - b \cdot e_1^2, a_{i0} \cdot a_{j0} - b \cdot a_{i1} \cdot a_{j1}, a_{i0} \cdot a_{j1} - a_{i1} \cdot a_{j0}\ (i, j = 1, 2, 3), J(2)\right)$$
$$\cap (a_{i0}, a_{i1}\ (i = 1, 2, 3)).$$

Using the interpretation of divisibility of polynomials, we find it easy to get information on the space $B(n)$ *as a set*.

THEOREM 4.2. *The space $B(n)$ has the following properties*:

(1) *There are $n + 1$ irreducible components $Y_0, \dots, Y_n$ and* $\dim Y_k = 2n - 1 + 2k$.

(2) *The normalization of Y_k is smooth*, $k = 0, \dots, n$.

(3) *The multiplicity of Y_k is* $\binom{n}{k}$.

Proof. The $\mathbf{C}$-valued points of $B(n)$ correspond exactly to choices of $a_{ij}, b_k, e_l \in \mathbf{C}$, such that the corresponding polynomials a_i, e and b have the property that the polynomial $a_i \cdot e$ is divisible by the polynomial $x^n + b$. Now if e has a factor F of degree k with $x^n + b$ in common, then each of the a_i has to be divisible by $G := (x^n + b)/F$. Let Y_k $(k = 0, \dots, n)$ be the subspace of $B(n)$ such that the polynomial e has at least k roots in common with the polynomial $x^n + b$ and the polynomials a_i $(i = 1, 2, 3)$ have at least $n - k$ roots in common with the polynomial $x^n + b$, the Y_k's are algebraic sets (which we give the reduced structure) and clearly $B(n) = \bigcup Y_k$ as sets. To describe the normalization of Y_k we do the following: define generic polynomials F and G:

$$F = x^k + \sum_{i=0}^{k-1} f_i x^i; \qquad G = x^{n-k} + \sum_{i=0}^{n-k-1} g_i x^i,$$

and consider the ring $S_k := \mathbf{C}[\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{b}, \underline{e}, \underline{f}, \underline{g}]$. Consider division with remainder in $S_k[x]$ through F and G and write

$$e = Q_F \cdot F + R_F(e), \qquad \deg(R_F(e)) < k, \quad \text{and}$$

$$a_i = Q_G(a_i) \cdot G + R_G(a_i), \qquad \deg(R_G(a_i)) < n - k, \quad i = 1, 2, 3.$$

Let L_k be the ideal in S_k generated by the coefficients of the polynomials $R_F(e)$, $R_G(a_i)$ and $F \cdot G - (x^n + b)$. The following facts are easy to check:

* There is a finite ring map

$$\mathbf{C}[\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{b}, \underline{e}]/J(n) = R/J(n) \to S_k/L_k$$

which sends each variable of R to the one in S_k with the same name.

* $S_k/L_k \approx \mathbf{C}[e_k, \dots, e_{n-1}, a_{in-k}, \dots, a_{in-1}, f_0, \dots, f_{k-2}, \underline{g}]$.

* The map $\tilde{Y}_k := \operatorname{Spec}(S_k/L_k) \to B(n)$ maps generically one-to-one onto the set Y_k.

The conclusion is that $\tilde{Y}_k$ is the normalization of Y_k and this proves statements (1) and (2) of the theorem. Statement (3) follows from the fact that Y_k is irreducible and that there are $\binom{n}{k}$ ways to choose k roots out of the n roots of $x^n + b$. We leave the details to the reader. ⊠

Remark 4.3. The equations of the space $B(n)$ are *linear* in the coefficients of the polynomials e and a_i, $i = 1, 2, 3$; so there are matrices M and N such that the equations can be written in either of the following two ways:

$$(a_{1,0}, \dots, a_{i,n-1}) \cdot M = 0, \qquad i = 1, 2, 3, \quad \text{or}$$

$$(e_0, \dots, e_{n-1}) \cdot N = 0.$$

Here M is the $n \times n$ matrix of "multiplication with e in the ring $R[x]/x^n + b$" and depends only on the e_l and b_k. Similarly, N is an $n \times (3n)$ matrix depending only on the a_{ij} and b_k, composed of the multiplication matrices of a_1, a_2 and a_3.

We define the ideals J_k $(k = 0, \dots, n)$ as the ideals generated by the $(k+1) \times (k+1)$ minors of M, the $(n+1-k) \times (n+1-k)$ minors of N, together with the equations of the space $B(n)$.

We can prove the following:

(1) The locus defined by J_k is Y_k.

(2) The ideals J_k are generically reduced.

(3) $J(n) = J_0 \cap \cdots \cap J_n$.

We have been unable to prove, however, that the J_k's are radical. When they are, (3) is the primary decomposition of $J(n)$.

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